Integrals: Crash Course Physics #3

Crash Course: Physics
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===== Introduction (0:00) =====

If you watched our last episode -and really, if you haven't, you should -you now know all about derivatives, and how to use them, to describe the way an equation is changing. Which means that now we can talk about the other main part of calculus, basically the inverse of derivatives, called integrals.

Integrals are useful because they also tell you a lot about an equation. If you plot an equation on a graph, the integral is equal to the area between the curve and the horizontal axis. Finding an integral is a little less straightforward than finding a derivative, but, as with derivatives, there are shortcuts we can use to make things a little easier. We'll even be able to use integrals to talk about how things move, specifically, the equation we've been calling the displacement curve, and why it looks the way it does.

(Crash Course Physics Intro)

===== Body (0:52) =====
Say you want to know how high your bedroom window is above the ground outside below. But you don't have anything to measure it with, just a ball, the stopwatch app on your phone, and your impressive knowledge of physics. The force of gravity is what makes the ball fall, so you that its acceleration is small g, 9.81 meters per second squared, downward. But you're trying to find its change in position, how far it falls.

We've spent a lot of time talking about the connection between qualities of motion; position, velocity, and acceleration. But so far, we've been describing that connection in a particular direction; velocity is the derivative of position, that is a measure of its change, and acceleration is the derivative of velocity. To figure out how far the ball falls, you need to use the reverse of that connection. Expressed mathematically, that means that velocity is the integral of acceleration, and position is the integral of velocity.

In other words, if you draw these equations on a graph, velocity is equal to the area under the acceleration curve, and position is equal to the area under the velocity curve. Finding that area is the tricky part.

There are simple ways to find the area of pretty much any shape, as long as the shape is made up nothing but straight lines and corners. And when you think about it, a curve is just a set of infinitely tiny lines. So the area under a curve can be divided into a set of infinitely tiny rectangles. Integrals tell you what happens when you divide the area under a curve into those infinitely many tiny rectangles, take the area of each of them, and add up those areas. So, how do you find that integral?

Well, you start by using the fact that integrals are basically the opposite of derivatives. If you know that your velocity is equal to twice time, for example, then you that's the derivative of the position. So, to find the equation for your position, you just have to look for an equation whose derivative is 2t, like x equals t-squared [$x=t^2$], for example.

It's kind of a roundabout way of doing things, compared to the neat equation we were able to use to find derivatives. But there is no tidy equation that we can use to calculate any integral we want. But, hold on don't panic! As with derivatives, there are shortcuts for finding certain, useful ones.

For instance, you can take the power rule that we used for derivatives, and run it backward. Basically, you add one to the exponent, then divide the variable by that number. So the integral of 2t, which is written like this, becomes t^2. In the same way, the integral of $42t^5$ is $7t^6$. You can take the trigonometric derivatives that we talked about, and do those backward, too. The integral of cos(x) is sin(x), and so on. And the integral of e^x is just e^x.

But there's one complication that we haven't talked about yet, maybe you've already spotted it, constants. A constant is basically just a number. It can literally be a number, like 2, or a half, or negative 4. Or it can be a placeholder for a number, like small g we've been using to represent the acceleration caused by gravity. And constants cause a problem when it comes to integrals because the derivative of a constant is just 0. A derivative is a rate of change, after all, so a constant, by definition doesn't change, and so will always have a derivative of zero. Which means that lots of different equations, an infinite number in fact, can all have the same derivative.

Like, the derivative of t^2 is 2t. But you can add any number, or a letter representing a number, to it and the derivative will still be 2t. So the derivative of t^2+1 is also 2t. And the same is true for t^2-7. Which means, if you're looking for the integral of an equation like x=2t, you have infinite choices, all of which are equally correct. t^2 would work, but so would t^2+1, or t^2-7, or $t^2+0.256$.

In these cases, we might know what shape of the integral should look like on a graph, like whether if it's a straight line or how it curves, but we don't know where to put it along the vertical axis. So we need to know what the constant is in order to know where to start drawing its integral. Whatever the constant is equal to, that's where the curve will intersect with the vertical axis. So t^2 would intersect at 0, but t^2-7 would intersect at negative 7. You get the idea.

Mathematicians had to figure out how to get around the problem of having infinite integrals to choose from, so they came up with a way to represent all of them; just add a C at the end of the integral. That C stands for all of the constants that we know we could put there. So if we say that the integral of 2t is t^2+C, then we're including t^2+1 and t^2-7 and all those other infinite options, every equation whose derivative is 2t.

But sometimes you don't need C at all, because you can figure out where your integral is supposed to be on the y axis, like if you have what's known as the initial value. In the case of a position graph, for instance, the initial value would be where you started out, so you'd know to draw the rest of the graph's shape from there. If you started at the 2 meter mark, say, and moved one meter every second, you'd put the graph here. But if you started at the 4 meter mark, you'd shift it up a little. Basically, it gives you the point where your integral intersects the vertical axis, which is the value of C.

Let's try it, and at the same time, we might as well figure out the height of your bedroom window. You're standing in your room, holding a tennis ball out the window with your arm resting on the sill. Now you drop the ball and start your stopwatch app at the same time. Turns out that the ball takes 1.7 seconds to hit the ground. Like we said earlier, we know the ball's acceleration, 9.81 m/s^2, and we know the time involved. Somehow, we have to get from there to the equation for the ball's position. So, first, let's find its velocity, the middle step, by taking the integral of its acceleration. Take a look at this graph of the ball's acceleration over time. It's just a flat line, which means that it should be pretty easy to find the area between it and the horizontal axis. It's a rectangle! And the area of a rectangle is just its base times its height. In this case, the base is t, the amount of time the ball took to fall. And the height is a, the acceleration. So the area between the acceleration graph and the horizontal axis is just (a) times (t). And the integral is (a) times (t) plus that constant we add, C. For now, we need the C, because we know the general shape of the velocity graph: It's a diagonal line slanted in such a way that every second, it rises by an amount equal to the acceleration. But we still don't know where to

PUT that line on the vertical axis. Not yet, anyway.

Now, we could have figured out the integral of acceleration just as easily by using the power rule: The acceleration, a, is a constant, but we could also say that it's (a) x (t^0), because anything raised to the power of 0 is just 1. So, according to the power rule, the integral of acceleration, which is the velocity, would be equal to the acceleration, multiplied by time, plus C. That's the same answer we got earlier!

Now, here's where the initial value comes in. The velocity graph tells you what the velocity is for each moment in time. But we had to add the C, because we didn't know where to place it on the vertical axis, when time equals zero. So the integral of the acceleration COULD have just been (acceleration)*(time), or a*t. But it could also have been (at) + 4. Or -6. So we put a C in the integral instead, to represent all those options. But we can get rid of that C if we can figure out the velocity, when time equals zero, what we've been calling v_0. And if we write our equation with that v_0 in it, as a placeholder for the velocity when time equals zero, we end up with the full equation for velocity. That should look familiar, because it's one of our kinematic equations: the definition of acceleration! Neat how everything works out like that. This equation tells us that the final velocity of our falling tennis ball, when it hit the ground, was 16.7 m/s downward.

But we aren't done yet. We're looking to link acceleration and POSITION, so we'll need to go one step further by integrating again. There are a couple of different ways we could do it, but let's just use the power rule again. The integral of (a*t) is (1/2)at^2, and the integral of v_0 is just v_0*t. Put 'em together, and you end up with this, which is starting to look a whole lot like ANOTHER kinematic equation, the one we've been calling the displacement curve. Now what about that C? Well, just like before with the starting velocity, the starting position will tell us where to stick this equation on the vertical axis. So we'll just make C equal to the starting position, which we'll write as x_0. And that's our integral—the displacement curve equation. Which means that now, we have everything we need to figure out how high your window is. The starting velocity is zero, because you just dropped the ball without throwing it. The acceleration is 9.81 m/s^2, and it took 1.7 seconds to land.

===== Outro (9:08) =====
And now you have everything there is to know about calculus... No you don't. As you can probably imagine, we've barely scratched the surface here. There's a reason it normally takes two semesters of university just to cover the basics. And, you know, some people spend their whole lives studying this stuff. But we've at least established enough background that when those things do come up in this course, we'll be able to use what we've covered here to talk about them. Today you learned that integrals are the area between an equation on a graph and the horizontal axis. You also learned a few shortcuts to help find them, and how to find C using an initial value.

===== Credits (9:38) =====

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